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DOMAIN BOUNDARIES, GOLDSTONE BOSONS

AND GRAVITATIONAL WAVES

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Abstract
The dynamical behaviour of domain boundaries between different realizations of the vacuum of scalar fields with spontaneously broken phases is investigated. They correspond to zero-modes of the Goldstone fields, moving with the speed of light, and turn out to be accompanied by strongly oscillating gravitational fields. In certain space-time topologies this leads to a quantization condition for the symmetry breaking scale in terms of the Planck mass.

1. Spontaneous symmetry breaking and domain boundaries

Spontaneous violation of rigid continuous symmetries in a field theory implies the existence of massless scalar excitations [1, 2]. A simple example is provided by the $U(1)$ invariant scalar model

$$\mathcal{L} = -\partial_\mu \varphi^* \partial^\mu \varphi - \frac{\lambda}{4} \left(\varphi^* \varphi - \mu^2 \right)^2 . \tag{1}$$

If $\mu^2 > 0$ we can parametrize the classical minima of the potential as

$$\varphi = \mu \, e^{i\theta}. \tag{2}$$

The angle θ is an unobservable parameter. However, if θ differs between various locations in space-time, the local variations of θ are observable. Indeed, as is well-known allowing the angle to become a space-time dependent field $\theta(x) = \sigma(x)/\mu\sqrt{2}$, the regular excitations of this field correspond to free, massless scalar bosons:

$$\mathcal{L} = -\frac{1}{2} \, \partial_\mu \sigma \partial^\mu \sigma, \tag{3}$$

with the associated field equation

$$\Box \, \sigma = 0. \tag{4}$$

Even if the scalar field is frozen in a classical minimum (2) over finite, extended regions of space-time, the dynamics of the fields can still cause the value of θ to differ in different regions. Such a situation is conceivable in cosmological models of the large-scale structure of the universe [3].

The boundaries between such regions correspond to zero-modes of the field equation (4):

$$\sigma(x) = \sigma_0 + \mu\sqrt{2} \, k \cdot x, \qquad k_\mu^2 = 0. \tag{5}$$

In the simplest case we have a flat boundary of finite thickness L, for example in the (y, z)-plane, with on one side ($x = +\infty$) a classical vacuum field configuration with arbitrary constant angle θ_0, and on the other ($x = -\infty$) a similar configuration with constant angle θ_1. Then the solution in the boundary region is

$$\sigma(x) = \sigma_0 + \mu\sqrt{2} \, k \, (x - t) \, , \qquad -L \leq (x - t) \leq 0. \tag{6}$$

Such a boundary corresponds to a right-moving collective excitation, moving at the speed of light[1]. The value of the wave number k is

[1] In natural units $c = 1$. Of course there is an equally valid solution moving to the left.

$$k = \frac{\theta_1 - \theta_0}{L} = \frac{\sigma_1 - \sigma_0}{\mu L\sqrt{2}}, \tag{7}$$

and the energy density in the wave $\mathcal{E} = 2\mu^2 k^2$.

The Noether current for the $U(1)$ symmetry of $\mathcal{L}$ is

$$j_\mu = -\frac{i}{2}\varphi^* \overleftrightarrow{\partial}_\mu \varphi = \frac{\partial_\mu \sigma}{\mu\sqrt{2}}. \tag{8}$$

It is conserved by the equation of motion (4).

2. Coupling to gravity

In cosmological applications, the Goldstone fields have to be coupled to gravity. The action becomes

$$S = \int d^4x \sqrt{-g}\left(\frac{-1}{16\pi G}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right). \tag{9}$$

It turns out, that the solutions (5) remain solutions also in the presence of gravity. However, the finite energy density $\mathcal{E}$ of the boundary moving at the speed of light now becomes a source of gravitational waves. These waves correspond to oscillations of the gravitational field which travel together with the boundary at the speed of light and remain associated with it. In fact, the solution we present below, in which an oscillating gravitational field is associated with a constant Goldstone current, has certain similarities with the Josephson effect in the theory of superconductivity.

For the system (9) the Einstein equations take the form

$$R_{\mu\nu} = -8\pi G\partial_\mu\sigma\partial_\nu\sigma, \tag{10}$$

whilst the scalar field equation becomes

$$\Box^{cov}\sigma = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\sigma = 0. \tag{11}$$

For the solutions (6) to remain valid, we make the following *Ansatz* for the metric

$$ds^2 = -dt^2 + dx^2 + f^2dy^2 + g^2dz^2, \tag{12}$$

with the additional restriction that $f = f(x-t)$ and $g = g(x-t)$. At this point the restriction is actually stronger than necessary, but this form is also required by the Einstein equations; therefore we impose it already here.

With the metric (12) the Einstein equations reduce to a single additional constraint

$$\frac{f''}{f} + \frac{g''}{g} = -16\pi G\mu^2 k^2. \tag{13}$$

Up to trivial relabeling of co-ordinates, this equation admits two families of solutions. If we take $f = g = 1$ for $x - t \geq 0$, then inside the boundary region $-L \leq (x - t) \leq 0$ the first class of solutions is

$$f = \cos\kappa\,(x - t)\,, \qquad g = \cos\lambda\,(x - t)\,. \tag{14}$$

The parameters (κ, λ) are related by

$$\kappa^2 + \lambda^2 = 16\pi G\mu^2 k^2. \tag{15}$$

These solutions will be called elliptic. Note that the solutions are continuous and differentiable at $x - t = 0$. They are generalizations of the plane fronted waves discussed for example in [4, 5]. The solutions can also remain continuous and differentiable at $x - t = -L$, provided for $x - t \leq -L$ we take

$$\begin{aligned} f(x-t) &= \cos\kappa L + \kappa\,(L + x - t)\,\sin\kappa L, \\ g(x-t) &= \cos\lambda L + \lambda\,(L + x - t)\,\sin\lambda L. \end{aligned} \tag{16}$$

Such solutions linear in $(x - t)$ correspond to flat space-time, because all components of the Riemann tensor vanish.

Clearly, for $\theta_1 \to \theta_0$ one has $k \to 0$ and as a consequence $(\kappa, \lambda) \to 0$. Then space-time becomes flat everywhere: $(f, g) \to 1$. This is not true in the other class of solutions, which are of the type

$$f = \cos\kappa\,(x - t)\,, \qquad g = \cosh\lambda\,(x - t)\,, \tag{17}$$

with the additional constraint

$$\kappa^2 - \lambda^2 = 16\pi G\mu^2 k^2. \tag{18}$$

These solutions we call hyperbolic. In this case the flat space behind the wave is parametrized as

$$\begin{aligned} f(x-t) &= \cos\kappa L + \kappa\,(L + x - t)\,\sin\kappa L, \\ g(x-t) &= \cosh\lambda L - \lambda\,(L + x - t)\,\sinh\lambda L. \end{aligned} \tag{19}$$

In the limit $k \to 0$ a plane fronted purely gravitational wave remains, with $\kappa = \lambda$.

3. Compactification and the quantization of the symmetry breaking scale

An interesting consequence of the previous results is, that if space-time can be compactified on a cylinder such that the Minkowski spaces at $x = \pm\infty$ can be identified, this requires a quantization condition

$$\kappa = \frac{2\pi n}{L}, \qquad \lambda = \frac{2\pi m}{L}, \tag{20}$$

for the elliptic solutions, and

$$\kappa = \frac{2\pi n}{L}, \qquad \lambda = 0, \tag{21}$$

for the hyperbolic solutions. At the same time, the scalar phase angles at $x = \pm\infty$ must be equal mod 2π as well; equivalently:

$$k = \frac{\theta_1 - \theta_0}{L} = \frac{2\pi l}{L}. \tag{22}$$

As a result we get for the elliptic case

$$16\pi G\mu^2 = \frac{n^2 + m^2}{l^2}. \tag{23}$$

By taking $m = 0$ this formula holds for the hyperbolic solutions as well. Thus a relation between Newton's constant and the symmetry breaking scale is obtained in a natural way. It can also be expressed in terms of the Planck mass as

$$\mu^2 = \frac{n^2 + m^2}{l^2} \frac{M_{Pl}^2}{16\pi}. \tag{24}$$

4. Orbits of test masses

In principle the oscillations of the metric components in equations (14) and (17) can be observed from the behaviour of test masses in the laboratory. To show this, we introduce a laboratory co-ordinate frame X^μ by defining

$$\begin{aligned} T &= t - \frac{\Lambda}{2}, \quad X = x - \frac{\Lambda}{2}, \\ Y &= fy, \qquad Z = gz, \end{aligned} \tag{25}$$

where

$$\Lambda = y^2 ff' + z^2 gg' = \frac{f'}{f} Y^2 + \frac{g'}{g} Z^2. \tag{26}$$

Moreover, it is useful to introduce light-cone co-ordinates in both frames by taking

$$U = X - T = u, \qquad V = X + T = v - \Lambda. \tag{27}$$

Then the line element (12) becomes

$$ds^2 = dUdV + dY^2 + dZ^2 + \left(\frac{f''}{f} Y^2 + \frac{g''}{g} Z^2 \right) dU^2 \tag{28}$$

Clearly, in the regions of constant scalar phase: $x - t \geq 0$ or $x - t \leq -L$, the co-efficient of dU^2 vanishes and we have manifestly Minkowskian regions of space-time. In the boundary region $-L < x - t < 0$ the line element reads explicitly

$$ds^2 = dUdV + dY^2 + dZ^2 - (\kappa^2 Y^2 \pm \lambda^2 Z^2)\, dU^2, \tag{29}$$

with the plus/minus sign corresponding to elliptic and hyperbolic solutions, respectively. In these co-ordinates the Riemann tensor is constant:

$$R_{UYUY} = -\kappa^2, \qquad R_{UZUZ} = \mp\lambda^2, \tag{30}$$

whilst all other independent components vanish. The metric does not oscillate, but test particles do with respect to the co-ordinate frame. This can be seen from the explicit solutions of the equations of motion

$$\frac{d^2 X^\mu}{d\tau^2} + \Gamma_{\lambda\nu}{}^{\mu} \frac{dX^\lambda}{d\tau} \frac{dX^\nu}{d\tau} = 0. \tag{31}$$

Using the line element (29) and choosing the unit of co-ordinate time in the origin $(Y_0 = Z_0 = 0)$ equal to the unit of proper time, the elliptic solutions read:

$$\begin{aligned} &U = U_0 - \tau, \qquad V = V_0 + \tau - \frac{1}{2} \left(\kappa Y_0^2 \sin 2\kappa\tau + \lambda Z_0^2 \sin 2\lambda\tau \right), \\ &Y = Y_0 \cos \kappa\tau, \quad Z = Z_0 \cos \lambda\tau. \end{aligned} \tag{32}$$

For the hyperbolic solutions we replace $\cos \lambda\tau$ by $\cosh \lambda\tau$, and $\sin 2\lambda\tau$ by $-\sinh 2\lambda\tau$. The oscillatory motion of the test masses with respect to the origin $Y = Z = 0$ is manifest in this laboratory frame.

5. Discussion

We have shown that plane fronted gravitational waves are present in regions where Goldstone bosons have constant gradients. Physically such a situation is present in boundary regions between domains of constant σ. In these regions the original scalar field ϕ oscillates as well:

$$\phi(x) = \mu\, e^{ik\cdot x}. \tag{33}$$

The frequency of these scalar waves is much higher in general than that of the associated gravitational wave: $k \gg \kappa$, unless the vacuum expectation μ is of the order of the Planck mass. Therefore the frequency of the gravitational waves becomes appreciable only for very large values of k, or very steep changes in the value of the phase θ. This applies for very thin boundaries.

Finally we would like to point out that instead of (massless) scalar potentials with constant gradients, one might also consider massless vector fields with constant field strength (E, B). The generation of gravitational waves in constant electromagnetic fields has been discussed elsewhere [6].

References

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